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COMMENT

AB percolation on close-packed graphs

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Abstract. For a class of close-packed graphs, the AB percolation critical probability is equal to the classical site percolation critical probability of the related graph obtained by inserting next-nearest-neighbour bonds. As a consequence, infinite AB percolation occurs for many fully triangulated graphs.

1. Introduction

We consider a variant of the classical percolation model which was introduced independently by Mai and Halley (1980) as 'AB percolation' and Sevšek *et al* (1983) and Turban (1983) as 'antipercolation'. In the model, the sites of an infinite graph G are independently labelled A and B with probabilities p and $1-p$, respectively. Neighbouring sites which have opposite labels are connected by a bond, while neighbouring sites which have a common label are not. The object of study is the probability distribution of the size of clusters of sites which are connected by AB bonds. As in classical percolation, the first step is to determine if infinite clusters exist for various values of the parameter p. This question is more interesting for AB percolation than classical site percolation, since there are common graphs for which infinite clusters cannot exist for any value of $p \in [0, 1]$ for AB percolation.

Mai and Halley (1980) treated AB percolation as a model for gelation, letting the labels represent occupancy of the site by one of two reactants, which are present in proportions p and $1-p$. Their Monte Carlo simulation indicated that infinite AB clusters exist for $p \in [0.2145, 0.7855]$ on the triangular lattice. Halley (1983) discussed a more general class of models, called 'polychromatic percolation' and proved the non-existence of infinite AB clusters when $p = \frac{1}{2}$ on bipartite graphs with site percolation critical probability greater than $\frac{1}{2}$. Since one expects that the probability of existence of an infinite AB percolation cluster is largest when $p = \frac{1}{2}$, this suggests that infinite AB clusters cannot exist for any value of p on such graphs (but this has not been proved). Turban (1983) and Sevšek *et al* (1983) were motivated by the study of antiferromagnetism. They provided an argument for existence of infinite AB percolation on the triangular lattice which, although incorrect, suggested the method presented in this comment. Wilkinson (1987) considered two-parameter percolation on bipartite

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graphs as a model for gelation and noted that AB percolation on a bipartite graph is a special case of his model.

Other than the work of Halley (1983), mathematically rigorous results have been obtained only recently. Scheinerman and Wierman (1987) produced the first example of a two-dimensional periodic graph on which AB percolation occurs. Appel and Wierman (1987) proved that AB percolation is impossible on a class of bipartite graphs (including the square and hexagonal lattices), partially verifying a conjecture of Halley (1983). Wierman and Appel (1987) proved that infinite AB percolation clusters exist on the triangular lattice when $p \in [0.497, 0.503]$. This was improved to $p \in [0.4031, 0.5969]$ by Wierman (1988) by identifying the threshold for AB percolation on the triangular lattice as the site percolation critical probability of the triangular lattice with nearest- and next-nearest-neighbour bonds. The main theorem of this comment generalises this result. Wierman (1988) also presented a proof that infinite AB percolation occurs on any graph with site percolation critical probability less than $\frac{1}{2}$.

2. Definitions and results

A graph G consists of a countable set $V(G)$ of vertices and a countable set $E(G)$ of pairs of vertices, called edges. An assignment of a label, A or B, to each vertex of G is a configuration on G, i.e. a configuration is an element $\omega \in \{A, B\}^{V(G)}$ or equivalently a function $\omega: V(G) \to \{A, B\}$. The AB percolation model is a probability model on the sample space $\{A, B\}^{V(G)}$ of configurations on G, with probability measure P_p such that the labels of vertices of G are independent random variables with probability p of labelling each vertex A.

An edge of G is an AB *bond* if its endpoints have different labels. An AB *path* is a path in which all edges are AB bonds. (We will use the terms 'A path' and 'B path' to refer to paths with all vertices labelled A or B, respectively. At times, we will refer to the labels as colours and refer to an object as 'monochromatic' if all its vertices have a common label.) The AB *cluster* containing a vertex v, denoted W_v^{AB}, is the set of vertices which may be reached from v through an AB path. The number of vertices in W_v^{AB} is denoted by $|W_v^{AB}|$. Define the AB percolation probability by

$$\theta_v^{AB}(p, G) = P_p[|W_v^{AB}| = \infty].$$

Note that AB paths and AB clusters are unchanged if the labels of all vertices are changed, but the parameter of the model is changed from p to $1-p$. Thus

$$\theta_v^{AB}(p, G) = \theta_v^{AB}(1-p, G)$$

for all $p \in [0, 1]$, so the AB percolation probability function is symmetric about $\frac{1}{2}$.

While the value of $\theta_v^{AB}(p, G)$ may depend on the vertex v, the set of values of p for which $\theta_v^{AB}(p, G) > 0$ is independent of the choice of vertex if G is a connected graph. Thus, for a connected graph G and an arbitrary site v, we define the AB critical probability by

$$p_H^{AB}(G) = \inf\{p: \theta_v^{AB}(p, G) > 0\}$$

with the convention that, if the set on the right-hand side is empty, $p_H^{AB}(G) = +\infty$. (We denote the site percolation probability function of a graph G by $\theta_v(p, G)$. We will denote the classical site percolation critical probability by $p_C(G)$ if the results of Menshikov *et al* (1986) or Aizenman and Barsky (1987) apply, so that the common versions of critical probability are all equal. Otherwise, $p_H(G)$ denotes the critical probability for existence of infinite clusters with positive probability and $p_T(G)$ denotes the critical probability for infinite expected cluster size.) Note that we do not know that the AB percolation probability is monotone in p, so it may be possible that $\theta_v^{AB}(p, G)=0$ for a value $p \in (p_H^{AB}, \frac{1}{2}]$.

For any graph G, define the graph G_2 as follows. The vertex set of G_2 is the same as that of G. The edge set of G_2 contains an edge between each pair of vertices which may be connected by a path of two edges in G. In the case of fully triangulated graphs G, the graph G_2 contains edges between all neighbours and next-nearest neighbours. Since successive vertices on an AB path have alternating labels, if an infinite AB path exists on G, then there exists an infinite A path on G_2 and an infinite B path on G_2. The existence of such paths requires that both $p \geq p_H(G_2)$ and $1-p \geq p_H(G_2)$ hold. Therefore, we have the lower bound

$$p_H(G_2) \leq p_H^{AB}(G) \tag{1}$$

and the fact that if $p_H(G_2) > \frac{1}{2}$, then $\theta_v^{AB}(p, G)=0$ for all $p \in [0, 1]$, i.e. that AB percolation is impossible on G.

Define the class $\mathscr{C}$ of graphs G constructed as follows. Consider a partition of R^d, $d \geq 1$, into polyhedra, such that every compact set of R^d intersects only finitely many polyhedra. The vertex set of G is the set of vertices of polyhedra in the partition. The polyhedra in the partition will be called cells (to distinguish them from unions of these). Close-packing a cell C means inserting edges between all pairs of vertices of C. The edge set of G is the set of edges resulting from close-packing all cells in the partition. For $d=2$, $\mathscr{C}$ is essentially the class of matching graphs of planar mosaics (defined in Kesten (1982)). The difference arises from the fact that a mosaic permits multiple edges between a pair of vertices, which are irrelevant when treating site percolation and AB percolation.

The principal result of this paper may now be stated.

Theorem.

(a) If $G \in \mathscr{C}$, then $\{p \leq \frac{1}{2}: \theta_v^{AB}(p, G) > 0\} = \{p \leq \frac{1}{2}: \theta_v(p, G_2) > 0\}$.

(b) If $G \in \mathscr{C}$ and $p_H(G_2) < \frac{1}{2}$, then $p_H^{AB}(G) = p_H(G_2)$.

The interpretation of (a) is that infinite AB percolation clusters exist for exactly the same parameter values at which infinite site percolation clusters exist on G_2. The simpler consequence (b) says that the two models have equal critical probabilities if $p_H(G_2) < \frac{1}{2}$, but does not provide the information contained in (a) about the behaviour at the critical probability.

Note that the theorem does not require the graph G to satisfy any symmetry or periodicity conditions. However, application of the result to prove the existence of AB percolation on a given graph G requires that we show $p_H(G_2) < \frac{1}{2}$ or $\theta_v(\frac{1}{2}, G_2) > 0$.

To apply the results of Kesten (1982) regarding the critical probabilities of two-dimensional dual percolation models and strict inequalities between critical probabilities of related two-dimensional graphs, such regularity conditions are needed.

The proof of the theorem is given in § 3 and discussion of its application to show the existence of AB percolation on fully triangulated periodic graphs, and matching graphs of periodic planar graphs, are given in § 4.

3. Proof of the theorem

The method of proof has its origin in an argument of Sevšek *et al* (1983). They remarked that, on the triangular lattice, all the boundary sites of a site percolation cluster of A belong to a common AB cluster. This is actually incorrect, since the boundary of an infinite cluster is made up of finite length boundaries of compact regions of its complement. However, an argument of Wierman (1988) constructed an infinite AB cluster by connecting together AB clusters that follow the boundary of an A cluster in T_2. The present proof simplifies the argument and extends it to the class $\mathscr{C}$ of graphs.

Let G be a graph in $\mathscr{C}$. If there is no $p \leq \frac{1}{2}$ for which $\theta_v(p, G_2) > 0$, then $\theta_v^{AB}(p, G) = 0$ for all $p \in [0, 1]$, by the reasoning that proved (1). Thus, we take $p \leq \frac{1}{2}$ with $\theta_v(p, G_2) > 0$. Then, with probability one, there exists an infinite A cluster on G_2 for this value of p. Since $1-p \geq \frac{1}{2}$, we also have $\theta_v(1-p, G_2) > 0$, so there is positive probability that the vertex v is in an infinite B path β in G_2, when the parameter value is p.

We proceed by assigning labels to the cells of the partition generating G. Label a cell A if at least one of its vertices is labelled A, and similarly for B. Clearly, any cell which has both types of vertices on its perimeter is labelled both A and B. Since such a cell is close-packed in G, all its vertices are in a common AB cluster on G.

The infinite A cluster in G_2 corresponds to an infinite connected union of cells labelled A, which we will call the A region. Any two vertices u, v that are adjacent in the A cluster in G_2 are within a distance two of each other in G, so there is a vertex w which is adjacent to both u and v. This implies that w is on the perimeter of a cell containing u and on the perimeter of a cell containing v. Then both these cells are labelled A, due to u and v, and their union is connected through the vertex w.

The topological boundary of the A region consists of the set of faces (which are portions of hyperplanes), each of which separates a cell of the A region from a cell which is not in the A region. Since one of the cells containing the face is not in the A region, all vertices of that cell are labelled B, which implies that both cells are labelled B. Therefore, every cell in the A region with a face on the boundary of the A region is labelled both A and B and all its vertices are in a common AB cluster in G. Thus, there is an AB cluster containing every vertex in each component of the boundary of this infinite A region in G_2.

If there is a surface containing infinitely many vertices in the boundary of this infinite A region in G_2, there is an infinite AB cluster in G, by the previous paragraph.

If the boundary of the infinite A region in G_2 consists of a union of bounded surfaces, we use the infinite B path β in G_2 as follows. The path β runs alternately through the A region and its complement, and there is a connected set of cells labelled B corresponding to β. When β passes through the A region, the corresponding cells are labelled both A and B, so all the vertices of these cells are in a common AB cluster in G. Thus, starting from the first vertex of β that is in the A region, we can construct an infinite AB path by alternately following β through the A region, then following the boundary of the A region around a component of the complement of the A region to the point where β enters it again.

Thus, we have established part (a) of the theorem, from which (b) follows immediately.

4. Remarks and comments

(a) To use the theorem to establish that AB percolation exists on a graph $G \in \mathscr{C}$, it suffices to prove that $p_H(G_2) < \frac{1}{2}$.

For a two-dimensional graph $G \in \mathscr{C}$ there is a corresponding planar mosaic, which we denote by M. Subject to technical conditions involving periodicity and symmetry, Kesten's (1982) principal theorem (3.1) yields $p_C(G) + p_C(M) = 1$ and his theorem 10.2 yields $p_C(G) < p_C(M)$ if M is not fully triangulated. While this would imply the existence of AB percolation on G, since $p_C(G) \leq p_C(G) < \frac{1}{2}$, the last inequality already suffices by Wierman (1988). Our theorem may allow determination of a better upper bound for $p_H^{AB}(G)$ in these cases, however.

If G is a fully triangulated planar graph (and appropriate conditions are satisfied), then $p_C(G) = \frac{1}{2}$ by Kesten's theorem 3.1 and the result from Wierman (1988) does not establish existence of AB percolation on G. Constructing a graph G' from G by inserting (periodically) an edge between two non-adjacent vertices in two triangles sharing a common edge (producing a close-packed quadrilateral), one applies Kesten's theorem 10.2 to obtain $p_C(G') < \frac{1}{2}$. By the inclusion principle, $p_C(G_2) < p_C(G') < \frac{1}{2}$, so AB percolation exists on periodic fully triangulated graphs which have one axis of symmetry, provided the conditions of Kesten's theorems are satisfied.

(b) It is not true that AB percolation occurs on every fully triangulated graph. Van den Berg (1981) exhibited a non-periodic fully triangulated graph G with $p_H(G) = 1$. It is easily shown that $p_H(G_2) = 1$ also, so AB percolation is impossible on G.

(c) It is not true that $p_H^{AB}(G) = p_H(G_2)$ for periodic graphs in general. Recall that, by Appel and Wierman (1987), AB percolation does not occur on the square lattice. However, S_2 contains the matching lattice S^M of S, so $p_H(S_2) \leq p_C(S^M) \leq 0.4966$, where the last inequality follows from Toth (1985) and Kesten (1982), theorem 3.1.

(d) The upper bound for the AB percolation critical probability of the triangular lattice may be improved to $p_H^{AB}(T) \leq 2\sin(\pi/18) \approx 0.3473$. Observe that T_2 contains the matching lattice of the hexagonal lattice, denoted H^M, so $p_H^{AB}(T) = p_H(T_2) \leq p_C(H^M) = 1 - p_C(H)$. Since the site percolation critical probability is bounded below by the bond percolation critical probability, we have $1 - p_C(H) \leq 2\sin(\pi/18)$.

References

Aizenman M and Barsky D 1987 *Commun. Math. Phys.* **108** 489-526

Appel M J and Wierman J C 1987 *J. Phys. A: Math. Gen.* **20** 2527-31

Halley J W 1983 *Percolation Structures and Processes* (*Annals of the Israel Physical Society* **5**) ed G Deutcher, R Zaller and J Adler (Bristol: Adam Hilger) pp 323-51

Kesten H 1982 *Percolation Theory for Mathematicians* (Basle: Birkhauser)

Mai T and Halley J W 1980 *Ordering in Two Dimensions* (Amsterdam: North Holland) pp 369-71

Menshikov M V, Molchanov S A and Sidorenko A F 1986 *Theor. Prob., Math. Statist., Theor. Cybern.* **24** 53-110

Scheinerman E R and Wierman J C 1987 *J. Phys. A: Math. Gen.* **20** 1305-7

Sevšek F, Debierre J-M and Turban L 1983 *J. Phys. A: Math. Gen.* **16** 801-10
Toth B 1985 *Z. Wahrsch. Verw. Geb.* **69** 19-22
Turban L 1983 *J. Phys. A: Math. Gen.* **16** L15-7
Van den Berg J 1981 *J. Math. Phys.* **22** 152-7
Wierman J C 1988 *Proc. Semester on Combinatorics and Graph Theory* (Warsaw: Banach Centre Publications) submitted
Wierman J C and Appel M J 1987 *J. Phys. A: Math. Gen.* **20** 2533-7
Wilkinson M K 1987 *J. Phys. A: Math. Gen.* **20** 3011-8